On the Edge-graceful spectra of cycles with one chord and dumbbell graphs

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ABSTRACT Let G be a (p, q)-graph and k≥0. A graph G is said to be k-edge graceful if the edges can be labeled by k, k+1,..., k+q-1 so that the vertex sums are distinct, mod p. We denote the set of all k such that G is k-edge graceful by egl(G). The set is called the edge-graceful spectrum of G. In this paper the problem of which sets of natural numbers are the edge-graceful spectra of two types of (p, p+1)-graphs is studied.

1. Introduction. Given an integer k ≥0, a graph G = (V, E) with p vertices and q edges is said to be k-edge graceful if there is a bijection f : E →{k, k+1, k+2,..., k+q-1} such that the induced mapping f+ : V →Zp, given by f+(u) =Σ{f(u,v): (u,v) in E} (mod p) is a bijection.

Figure 1 shows that K4 is k-edge graceful for k =1, 2, 3, 4.

The study of 1-edge-graceful graphs was initiated by S.P. Lo [17]. Edge-graceful labeling can be viewed as the dual concept of graceful labeling. Lee, Lee, Murthy [5] showed that if G is a (p, q)-graph with p ≡ 2 (mod 4) then G is not 1-edge-graceful.

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We have the following necessary condition for the k-edge graceful graph which is a generalization of Lo’s condition in [17]

**Theorem 1.** If a (p,q)-graph G is k-edge-graceful then it satisfies the condition

\[ q(q+2k-1) = \frac{p(p-1)}{2} \pmod{p} \]

The theory of 1-edge graceful graphs is completely different from other k-edge graceful graphs. For example, trees of order 4 are 2-edge graceful but not 1-edge-graceful. (Figure 2)

In this paper we consider connected (p,p+1)-graphs. We consider two types of k-edge graceful (p,p+1) graphs: cycle with a chord and dumbbell graph D(a,b) where a+b=p. Assume the vertices of cycle are \(v_1, v_2, \ldots, v_p\) and the chord connect vertex \(v_i\) with \(v_j\). We denote this graph by \(C_p(r)\). The dumbbell graph D(a,b) is formed by join two disconnected cycles \(C_a\) and \(C_b\) by an edge (Figure 3).
We denote the set of all integers $k > 0$ such that $G$ is $k$-edge graceful by $\text{egl}(G)$. The set is called the edge-graceful spectrum of $G$. In this paper the problem of which sets of natural numbers are the edge-graceful spectra of $(p,p+1)$-graphs is studied.

1-edge-graceful graphs are investigated in [1,3,4,5,6,7,8,9,10,11,12,13, 14,18,19,20,21,22,23,24]. Some $k$-edge graceful graphs are considered in [15,16]. A good account on other graph labeling problems can be found in the dynamic survey of Gallian [2].

2. Edge-graceful Index spectrum of a cycle with a chord.

We denote the set of natural numbers by $N$.

By Theorem 1, a necessary condition for the $(p,p+1)$-graph $G$ to be $k$-edge-graceful is

$$(p+1)(p+2k) \equiv p(p-1)/2 \pmod{p} \implies 2k \equiv (p-1)/2 \pmod{p}$$

Furthermore, consider the condition for each case as follows:

Case 1. $p = 2s + 1$: $2k \equiv s(2s+1) \pmod{2s+1} \implies 2k \equiv 0 \pmod{2s+1}$

$\implies k \equiv 0 \pmod{p}$

Case 2. $p = 2s$: $2k \equiv s(2s-1) \pmod{2s} \implies 2k \equiv -s \pmod{2s}$

$\implies 2k \equiv s \pmod{2s}$

(1) odd $s$: there is no $k$ satisfy the congruence

(2) even $s$: Let $s = 2t$, then $2k \equiv 2t \pmod{4t}$

Thus $k \equiv t \pmod{2t}$, i.e. $k \equiv s/2 \pmod{s} \implies k \equiv p/4 \pmod{p/2}$

Summarizing the observations above, we propose the following conjecture

**Conjecture.** Let $G$ be the $(p,p+1)$-graph

(A) when $p$ is odd, $\text{egl}(G) = \{sp: s=0,1,2,\ldots}\}$

(B) when $p$ is even

and (1) $p \equiv 2 \pmod{4}$, $\text{egl}(G) = \emptyset$.

(2) $p = 4t$, $\text{egl}(G) = \{np/2+p/4: n=0,1,2,\ldots\}$

Now we want to show that the conjecture is true for the cycle with one chord for odd order.
**Theorem 2.** If $G$ is a cycle with one chord of odd order $p$, then $\text{egl}(G) = \{\text{sp}: s = 0, 1, 2, 3, \ldots\}$

**Proof.** It suffices to show that $C_p(r)$ is 0-edge-graceful for any $r > 2$. We label the chord with 0, and label the edges of a cycle clockwise consecutively by 1, 2, ..., $p$. Then we see that the vertices have label 1, 3, 5, ..., 0, 2, 4, ..., $p - 1$. It is clear that the label is 0-edge-graceful (Figure 4).

If we add each edge-label by $sp$, we can have a $sp$-edge-graceful labeling. Thus $\text{egl}(C_p(r)) = \{\text{sp}: s = 0, 1, 2, 3, \ldots\}$.

**Example 1.**

![Figure 4.](image)

**Example 2.** Figure 5 shows that $C_4(3)$ is 1-edge-graceful and 3-edge-graceful. For integer $n \geq 1$, if we add $4n$ to each edge label then we can show that it become $4n + 1$-edge-graceful and $4n + 3$-edge-graceful. Thus we conclude the edge-graceful spectrum of $C_4(3)$ is $\text{egl}(C_4(3)) = \{1, 3, 5, 7, 9, \ldots, 2n + 1, \ldots\}$.

![Figure 5.](image)

$C_4(3)$ is 1-edge-graceful $\quad$ $C_4(3)$ is 3-edge-graceful

**Example 3.** Figures 6 shows that $C_8(4)$ is 2-edge-graceful with two different labelings and Figure 7 demonstrates that it is 6-edge-graceful.
$C_8(4)$ is 2-edge-graceful

$C_8(4)$ is 2-edge-graceful

Figure 6.

$C_8(4)$ is 6-edge-graceful

Figure 7

**Example 4.** Figures 8 shows that $C_8(5)$ is 2-edge-graceful and 6-edge-graceful.

$C_8(5)$ is 2-edge-graceful

$C_8(5)$ is 6-edge-graceful

Figure 8.

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Example 5.

\[ C_{12}(7) \text{ is 3-edge-graceful} \quad \text{and} \quad C_{12}(7) \text{ is 9-edge-graceful} \]

**Figure 9**

In general, it is difficult to find the k-edge graceful labeling for even cycle with a chord. At present, we can solve the following type of even cycle with a chord.

**Theorem 3.** The cycle with one chord \( C_{4n}(n+1) \) has edge-graceful spectrum \( \text{egl}(C_{4n}(n+1)) = \{ n+4nt, 3n+4nt: t=0,1,2,\ldots \} \)

**Proof.** We want to show that the cycle with one chord \( C_{4n}(n+1) \) is n-edge graceful and 3n-edge-graceful.

For the case of n-edge-graceful labeling, we label the edges of the cycle begin from \( (v_1, v_2), (v_3, v_4), (v_5, v_6), \ldots, (v_{4n-1}, v_{4n}) \) clockwise by \( \{ n, n+1, n+2, \ldots, 3n-1 \} \). Next we label \( (v_2, v_3), (v_4, v_5), (v_6, v_7), \ldots, (v_{4n-2}, v_{4n-1}), (v_{4n}, v_1) \) clockwise by \( \{ 3n+1, 3n+2, \ldots, 5n \} \) and next to the edges already labelled.

We see that the above edge labeling will contribute vertices with labels from 1 to 4n-1 and 6n (\( \equiv 2n \mod 4n \)).

Now we label 3n to the chord which connects the two vertices \( v_1 \) and \( v_{n+1} \) which have label \( n \) and 6n (\( \equiv 2n \mod 4n \)) respectively. Now after we add this new edge label, we change the labelings of these two vertices from \( n \) to \( 4n (\equiv 0 \mod 4n) \) and from \( 6n (\equiv 2n \mod 4n) \) to \( 9n (\equiv n \mod 4n) \).

Figure 10 shows the labeling scheme for \( C_8(3) \) and \( C_{12}(4) \).
For the 3n-edge-graceful labeling case, the solution is almost the same as above. we label the edges of the cycle begin from $(v_{n+1}, v_{n+2}), (v_{n+3}, v_{n+4}), \ldots, (v_{4n-1}, v_{4n})$, $(v_1, v_2), (v_3, v_4), \ldots, (v_{n-1}, v_n)$ clockwise by $\{3n, 3n+1, 3n+2, \ldots, 5n-1\}$.

Next we label $(v_{n+2}, v_{n+3}), (v_{n+4}, v_{n+5}), \ldots, (v_{4n-2}, v_{4n-1}), (v_{4n}, v_1), (v_3, v_4), (v_5, v_4), \ldots, (v_3, v_4)$ clockwise by $\{5n+1, \ldots, 7n\}$.

We see that the above edge labeling will contribute vertices $\{v_{n+2}, v_{n+3}, v_{n+4}, \ldots, v_{4n-1}, v_{4n}, v_{1}, \ldots, v_{n}, v_{n+1}\}$ with labels in the order $\{1, 2, \ldots, 4n-2, 4n-1, 2n\}$.

Now we label 5n to the chord which connects the two vertices $v_1$ and $v_{n+1}$ which have label 3n and 2n respectively. Now after we add this new edge label 5n, we change the labelings of these two vertices from 3n to 8n($=0$ mod 4n) and from 2n to 7n($=3n$ mod 4n).

Figure 11 shows the labeling scheme for $C_8(3)$ and $C_{12}(4)$. □
3. Edge-graceful spectra of dumbbell graphs.

**Theorem 4.** For any odd \( m \geq 3 \), and even \( n \geq 4 \), the edge-graceful spectrum of the dumbbell graph \( D(m,n) \) is \( \text{egl}(G) = \{s(m+n): s=0,1,2,\ldots\} \)

**Proof.** It suffices to show that \( D(m,n) \) is 0-edge-graceful. We provide here two different 0-edge-graceful labelings:

**Method 1.** We start at the odd cycle \( C_m \) by labeling 0 to \( m-1 \) to each edge anticlockwise consecutively from one side of the connected vertex. Then we label \( m \) to the connected edge. Finally we label \( m+1 \) to \( m+n \) to each edge of the cycle \( C_n \) clockwise from one side of the connected vertex.

The vertices of \( C_m \) has induced labels \( \{1,3,5, 2m-1\} \) and the vertices of \( C_n \) has induced labels \( \{2m+1,\ldots,n+m= 0 \pmod{m+n}, 2,4,6,\ldots,m+n-1\} \)

**Method 2.** We start at the even cycle \( C_n \) by labeling 0 to \( n-1 \) to each edge clockwise consecutively from one side of the connected vertex. Then we label \( n \) to the connected edge. Finally we label \( n+1 \) to \( m+n \) to each edge of the cycle \( C_m \) anticlockwise from one side of the connected vertex.

The vertices of \( C_m \) has induced labels \( \{1,3,5, 2m-1\} \) and the vertices of \( C_n \) has induced labels \( \{2m+1,\ldots,n+m= 0 \pmod{m+n}, 2,4,6,\ldots,m+n-1\} \)

Now for each integer \( s \geq 1 \), we add \( (m+n)s \) to each edge label of the above edge-labeling we obtain a \( (m+n)s \)-edge-graceful labeling of \( D(m, n) \).

**Corollary 5.** For any integer \( n \geq 2 \), the edge-graceful spectrum of dumbbell graph \( D(2n,2n+1) \) is \( \{(4n+1)s: s=1,2,\ldots\} \)

We illustrate the above result for \( n=2 \).

**Example 6.**

![Diagram of D(4,5) is 0-edge-graceful](image)

**Figure 12.**

**Corollary 6.** For any integer \( n \geq 2 \), the edge-graceful spectrum of the dumbbell graph \( D(2n-1,2n) \) is \( \{(4n-1)s: s=1,2,\ldots\} \)

**Example 7.** We illustrate the above result for \( n=2 \) and \( 3 \) (Figure 13).
In particular, we have

**Corollary 7.** For any even integer \( n \geq 2 \), the edge-graceful spectrum of the dumbbell graph \( D(3,n) \) is \( \text{egl}(D(3,n))=\{s(3+n)\}: s=0,1,2,\ldots \} \)

**Example 8.** We provide here two different 0-edge-graceful labelings of \( D(3,4) \).

**Method 1**

![Method 1 diagram](image1)

\( D(3,4) \) is 0-edge-graceful

**Method 2**

![Method 2 diagram](image2)

\( D(3,4) \) is 0-edge-graceful

*Figure 13.*
Theorem 8. For any integer \( n \geq 2 \), the edge-graceful spectrum of dumbbell graph \( D(2n,2n) \) is \( \{2n s+n : s=0,1,2,\ldots\} \).

Proof. For convenience, we will index all vertices of the dumbbell graph \( D(2n,2n) \) by the following way.

For the two vertices which connects the two cycles \( C_{2n} \) by an edge, we index them by \( v_1 \) and \( v_{2n+1} \), respectively. The other vertices of the cycle \( C_{2n} \) which contains \( v_1 \) are indexed by \( v_2 \) to \( v_{2n} \) clockwise from the vertex next to \( v_1 \). The other vertices of the cycle \( C_{2n} \) which contains \( v_{2n+1} \) are indexed by \( v_{2n+2} \) to \( v_{4n} \) counterclockwise from the vertex next to \( v_{2n+1} \).

Now we are ready to give the edge labeling. Before that, we divide all edges into five sets: 

\[
S_1 = \{ (v_1, v_2), (v_2, v_1), (v_1, v_{2n+1}), (v_{2n+1}, v_{2n+2}), (v_{2n+2}, v_{2n+1}) \}, \\
S_2 = \{ (v_3, v_4), (v_5, v_6), \ldots, (v_{2n-3}, v_{2n-2}), (v_{2n-2}, v_{2n-1}) \}, \\
S_3 = \{ (v_2, v_3), (v_4, v_5), \ldots, (v_{2n-4}, v_{2n-3}), (v_{2n-3}, v_{2n-2}) \}, \\
S_4 = \{ (v_{2n+3}, v_{2n+4}), (v_{2n+4}, v_{2n+5}), \ldots, (v_{4n-3}, v_{4n-2}) \}, \\
S_5 = \{ (v_{2n+2}, v_{2n+3}), (v_{2n+4}, v_{2n+5}), \ldots, (v_{4n-4}, v_{4n-3}), (v_{4n-2}, v_{4n-1}) \}.
\]

Then we label \{n+1, n+2, \ldots, 2n-2, 2n-1\} to \( S_2 \) clockwise. We label \{3n+1, 3n+2, \ldots, 4n-2, 4n-1\} to \( S_1 \) clockwise. We label \{2n+1, 2n+2, \ldots, 3n-2, 3n-1\} to \( S_4 \) counterclockwise. We label \{4n+1, 4n+2, \ldots, 5n-2, 5n-1\} to \( S_5 \) counterclockwise. Finally for edges in \( S_3 \), we label \((v_1, v_2)\) by \( 5n \), \( (v_{2n}, v_1)\) by \( 4n \), \( (v_1, v_{2n+1})\) by \( 3n \), \( (v_{2n+1}, v_{2n+2})\) by \( 2n \), \( (v_{4n}, v_{2n+1})\) by \( n \).

We can see that such an edge labeling contribute vertices of the cycles \( C_{2n} \) containing \( v_1 \) from 0 to 2n-1, vertices of the cycles \( C_{2n} \) containing \( v_{2n+1} \) from 2n to 4n-1. □

Example 14. We have a 4-edge-graceful labeling of \( D(8,8) \). (Figure 15).

\[ \text{D}(8,8) \text{ is 4-edge-graceful} \]

Figure 15.

Finally, we want to point out there is an interesting connection between the cycle with a chord \( C_{4n}(2n+1) \) and the dumbbell graph \( D(2n,2n) \).
**Theorem 9.** An n-edge-graceful labeling of $C_{4n}(2n+1)$ can be obtained from a n-edge-graceful labeling of $D(2n, 2n)$.

**Example 15** The following figure 16 illustrate how to get a 7-edge-graceful labeling of $C_{28}(15)$ from a 7-edge-graceful labeling of $D(14, 14)$. The way is deleting the edges $(v_1, v_{14})$, $(v_{15}, v_{16})$, and add two edges $(v_{15}, v_{14})$, $(v_1, v_{16})$ with the original labeling of $(v_1, v_{14})$, $(v_{15}, v_{16})$ respectively. We observe the vertex labels of $v_1, v_{15}$ in $D(14, 14)$ is interchange in $C_{28}(15)$. The same result happens on the case of $(7+14k)$-edge-graceful labeling.

![Figure 16](image)

**Corollary 10.** For any integer $n \geq 2$, the edge-graceful spectrum of the cycle with a chord $C_{4n}(2n+1)$ is $\text{egl}(C_{4n}(n+1)) = \{ 2n s + n : s = 0, 1, 2, \ldots \}$

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References


